Binary Sequent Calculi for Finite Many-valued Logics

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Abstract. This paper presents a new *binary* sequent calculi for many valued logic with a finite set of truth values, in which these calculi are both sound and complete, and Kripke-like semantics for these calculi. In order to define a non-matrix based sequent calculus, based on the generalization of the classic 2-valued truth-invariance entailment, we transform many-valued logic into positive 2-valued multi-modal logic with classic conjunction, disjunction and a finite set of modal connectives. In this algebraic framework, we define a uniquely determined axiom system by extending the classic 2-valued distributive lattice logic by a new set of sequent axioms for many-valued logic connectives. Dually, in an *autoreferential* Kripke-style framework, we obtain a uniquely determined frame, where each possible world is an equivalence class of Lindenbaum algebra for a many-valued logic as well, represented by a truth value.

1 Introduction

A large number of real-world applications in Artificial Intelligence deal with partial, imprecise and uncertain information. In order to handle this kind of information, non-classic truth-functional many-valued logics such as fuzzy logic, bilattice-based logic, paraconsistent logic, etc. were introduced. The first formal semantics for modal logic was based on many-valuedness, proposed by Lukasiewicz in 1918-1920, and it was consolidated in 1953 in a 4-valued system of modal logic [1]. All cases of many-valued logics mentioned above are based on a lattice $(X, \leq)$ of truth values.
Sequent calculus, introduced by Gentzen [2] and Hertz [3] for classical logic, was generalized to the many-valued case by Rouseau [4] and others. The tableaux calculi were presented in [5,6]. The standard two-sides sequent calculus for lattice-based many-valued logics have been elaborated recently (with an autoreferential Kripke-style semantics for such logics) in two complementary ways [7,8,9,10].
In this research, we consider the truth-functional algebraic semantics for a given logic language, with a set of ground formulae (without variables) $\mathcal{L}$, with a set of many-valued logic connectives in Σ and predicate (or propositional) letters $p, q, r, ..$ A many-valued valuation $v : \mathcal{L} \rightarrow X$ in this research is a homomorphism between the free syntax-algebra of this logic language and the algebra (X, Σ) of truth-values. That is, for any n-ary logic connective $\odot$ in Σ, $\odot : X^n \rightarrow X$, we have $v(\odot(\phi_1, ..., \phi_n) = \odot(v(\phi_1), ..., v(\phi_n))$, where $\phi_i \in \mathcal{L}$, $1 \leq i \leq n$ are logic formulae in $\mathcal{L}$. Notice that a

non truth-functional many-valued semantics is also available [11]. It is based on "non-deterministic" connectives $\widetilde{\odot} : X^n \rightarrow \mathbf{2}^X$ (where A^B denotes the set of all functions from the set B to A and $\mathbf{2} = \{0, 1\}$ is a complete lattice of classical 2-valued logic) for each ordinary connective $\odot \in \Sigma$, such that $v(\odot(\phi_1, ..., \phi_n) \in \widetilde{\odot}(v(\phi_1), ..., v(\phi_n))$, and so called Nmatrices (non-deterministic matrices). However, in this "nondeterministic" case, the compositional property (the homomorphic property above) of a many-valued non-deterministic valuation v is not valid. Thus, in this research, we consider only the truth-functional many-valued logics where the valuations are homomorphic.

The well-known semantics of the many-valued logics is based on algebraic matrices (X, D) where $D \subset X = \{x_1, ..., x_m\}$ is a strict subset of designated truth values, so that a valuation v is a model for a formula $\phi \in \mathcal{L}$ iff $v(\phi) \in D$. This semantics is commonly used in practice, particularly when the number of truth-values is limited. As in the case of three-valued propositional logic, two different choices of the set of designated values for the same semantics respectively give Kleene logic and a basic paraconsistent logic J_3, both are very important for mathematical logic and its applications.

The well known sequent system developed for such a many-valued logic with matrix-based semantics is an ad hoc system based on m-sequents. More detailed information can be found in [12,13]. Although the approach to this well known m-sequent system is absolutely correct and useful, it has some minor drawbacks:

- It is not well-known when compared to those that are based on ordinary two-sided sequents. The framework for two-sided sequent calculi are well-understood and a lot of programs have been made in developing their efficient implementations.
- The use of two-sided sequents reflects the basic fact that logic is all about consequence relations. In the m-sequent calculus, only some characterization of the consequence relation can be done in a roundabout way.
- The use of two-sided sequents is universal and independent of any particular semantics, while the use of m-sequents relies on specific semantics for a given logic.

Consequently, it is interesting to consider a calculus for many-valued logics based on standard *binary sequents*. The previous work in this direction is recently proposed in [14] ant it is based on m-valued Nmatrices, signed formulae and a Rasiowa-Sikorski deduction system [15,16]. In the approach used in [14], an m-sequent calculus is transformed into the ordinary two-sided sequent system (where each sequent is of the form $\Gamma \vdash \triangle$, where $\Gamma, \triangle \subset \mathcal{L}$ are the finite subsets of logic formulae).

Notice that both sequent systems above are based on *matrix semantics* for logic entailment. This is not the case in our research: we use the *truth-invariance* semantics for many-valued logic [17] that is different from matrix-based semantics.

In this paper, we propose this new approach to the semantics of many-valued logics by transforming the original many-valued logic into the 2-valued multi-modal logic [18]. Then, based on the classical 2-valued distributive lattice logic (DLL) [19] extended by a set of new axioms for this 2-valued transformation of many-valued logic, we apply the Dunn's binary-sequent approach. In the original Dunn's approach, each sequent is of the simple form $\Phi \vdash \Psi$ (here Φ and Ψ are the 2-valued multi-modal formulae) and, consequently, his system is a particular sequent calculus where the left side of a sequent is not generally *a set* of 2-valued formulae but a single formula. Other differences between the previous approaches and the approach used in this research are as follows:

- We will not use the many-valued Rosser-Turquette operators J_k [20] that have been introduced long before the appearance of the Kripke semantics for algebraic modal operators. We will replace them by modal operators as follows: adopt the ontological encapsulation of many-valued logic into 2-valued multi-modal logic [18] with algebraic modal operators $[x] : X \rightarrow \mathbf{2}$ for any $x \in X$. We use $[x]$ both as a modal operator, i.e., as a syntactic language entity for modality "heaving a truth-value x" (it express exactly the *modality* of truth of the formula ϕ) and as a function on truth values, i.e., a semantic entity, which will be clear from the particular context where $[x]$ is used.
- We avoided using the signed formulae used in [14]. In this way, by using modal approach and its standard Kripke semantics, we are not obligated to develop an unnecessary ad-hoc semantics for such a calculus as shown in Section 4.

Notice that the main result of this work is that we obtained a standard *binary* sequent calculi for finite many-valued logic with the *truth-invariance semantics* of logic entailment. Consequently, this work is not only another new reformulation of the same many-valued inference system based on matrices but it is a substantially new inference system that is different from the m-sequents and from the sequent calculi in [14] (that are mutually equivalent). We apply this new approach to many-valued predicate logic (without quantifiers), with the set of k-ary predicate letters in P, and to its particular propositional case (where all predicate letters have 0-arity).
The rest of the paper is organized as follows: After a brief introduction to the truth-invariance inference semantics, multi-modal *predicate* logics (without quantifiers), and a short introduction to binary sequents and bivaluations, Section 2 presents the reduction of finite many-valued propositional (and predicate) logic language $\mathcal{L}$ into the 2-valued multi-modal algebraic logic language $\mathcal{L}_M$. After that, we show the main properties of this positive logic (with standard 2-valued conjunction and disjunction and a modal operator $[x]$ for each truth value x in a finite set X). We present normal-forms reduction as well, where each obtained formula has the modal operators applied only to propositional letters in $\mathcal{L}$ (atoms in predicate logic with Herbrand base H). Section 3 presents the binary sequent system $\mathcal{G}$ that we developed by extending the classic 2-valued DLL with the set of sequent axioms for each logic connective of a many-valued logic language $\mathcal{L}$. We show that this proof-theoretic sequent logic is sound and complete w.r.t. the model-theoretic semantics, based on many-valued valuations: each *deduced* sequent from $\mathcal{G}$ and a given set of sequent assumptions Γ is also a *valid* sequent (satisfied for every many-valued valuation) and vice versa. Finally, in Section 4, we present the development of an autoreferential Kripke-style semantics, based on Lindenbaum algebra of a many-valued logic language $\mathcal{L}$. We also define the Kripke frame for it with the set of possible worlds equal to the set of truth values X. After that, we show that the semantics is correct (sound and complete) for the multi-modal logic language $\mathcal{L}_M$, i.e., we demonstrate that for each many-valued algebraic model, we obtain a correspondent Kripke model, so that a formula that is true in $\mathcal{L}_M$ is true in this Kripke model as well and vice versa.

1.1 Truth-invariance model-theoretic entailment

In this paper, we denote the set of all functions from A to B by B^A, and a n-fold Cartesian product $A \times ... \times A$ for $n \geq 1$ by A^n, and the set of all subsets of A by $\mathcal{P}(A)$.
In the standard 2-valued model-theoretic semantics, we say that a valuation $v : \mathcal{L} \rightarrow \mathbf{2}$ is a *model* of a sentence $\psi \in \mathcal{L}$ iff $v(\psi) = 1$ (here $\mathcal{L}$ denotes the set of all ground formulae of a given logic language). Consequently, a formula ϕ is deduced from the set of formulae $\Gamma \subseteq \mathcal{L}$, denoted by $\Gamma \models_1 \phi$, iff $\forall v \in Mod_\Gamma.(v(\phi) = 1)$, where Mod_Γ is the set of all models of the formulae in Γ (here we use the index $1 \in \mathbf{2}$ in the consequence relation $\models_1$ to indicate the deduction of the *true* formulae).
The set Γ can be formally constructed by a subset of formulae Γ_1 that we want to be (always) true, and by a subset of formulae Γ_0 that we want to be (always) *false* so that $\Gamma = \Gamma_1 \bigcup \{\neg\phi \,|\phi \in \Gamma_0\ \}$, where $\neg$ is the 2-valued negation operator ($\neg 1 = 0, \neg 0 = 1$).
What is not often highlighted is that this standard 2-valued model-theoretic semantics *implicitly* defines the set of *false* sentences deduced from Γ as well, denoted here explicitly by the new derived symbol $\models_0$ for the deduction of false sentences (with the index $0 \in \mathbf{2}$) by: $\Gamma \models_0 \phi$ iff $\Gamma \models_1 \neg\phi$, that is, iff $\forall v \in Mod_\Gamma.(v(\phi) = 0)$.
Consequently, the classic 2-valued truth-invariance semantics of logic entailment can be paraphrased by the following generalized entailment, denoted by $\Gamma \models \phi$:
(CL) "a formula ϕ is a logic consequence of the set Γ" iff $(\exists x \in \mathbf{2})(\forall v \in Mod_\Gamma).(v(\phi) = x)$.
Thus, classic 2-valued entailment deduces both true and false sentences if they have the same (i.e., *invariant*) truth-value in all models of Γ. The consequence relation $\models_1$ defines Tarskian closure operator $C : \mathcal{P}(\mathcal{L}) \rightarrow \mathcal{P}(\mathcal{L})$ such that $C(\Gamma) = \{\phi \mid \phi \in \mathcal{L}$ and $\Gamma \models_1 \phi\}$. In the 2-valued logics we do not need to use the consequence relation $\models_0$ because the set of false sentences deduced from Γ is equal to the set $\{\neg\phi \mid \phi \in \mathcal{L}$ and $\Gamma \models_1 \phi\} = \{\neg\phi \mid \phi \in C(\Gamma)\}$. This particular property explains why, in the classic 2-valued logic, it is enough to consider only the consequence relation for deduction of true sentences, or alternatively the Tarskian closure operator C.
In the case of many-valued logics, it is not generally the case and we need the consequence relations for the derivation of sentences that are not true as well. Consequently, we will extend this classic 2-valued model-theoretic truth-invariance semantics of logic entailment to many-valued logics as well by:
(MV) $\Gamma \models \phi$ iff $(\exists x \in X)(\forall v \in Mod_\Gamma).(v(\phi) = x)$,
where Mod_Γ is specified by prefixing a particular truth-value $y \in X$ to each formula $\psi \in \Gamma$, as we explained previously in the case of the 2-valued logic with two subsets Γ_1 and Γ_0 for prefixed true and false sentences.
The matrix-based inference is different and specified by:
(MX) $\Gamma \models \phi$ iff $(\forall v \in Mod_{\Gamma,D}).(v(\phi) \in D)$,
with the set of models $Mod_{\Gamma,D}$ of Γ defined as the set of valuations v such that $(\forall\psi \in \Gamma).(v(\psi) \in \mathcal{D})$.
It is easy to verify that both semantics, the truth-invariance (MV) and the matrix-based (MX), in the case of classic 2-valued logics, where $D = \{1\}$, coincide.
This new truth-invariance semantics of logic entailment for many-valued logics has been presented for the first time in [17] and successively used for a new representation theorem for many-valued modal logics [10].

1.2 Introduction to multi-modal predicate logic

More exhaustive and formal introduction to modal logics and their Kripke models can be found in the literature [21]. Here, we provide only a short informal version, in order to make more clear definitions that are used in the next few paragraphs.
A predicate multi-modal logic, for a language with a set of predicate symbols $r \in P$ with arity $ar(r) \geq 0$ and a set of functional symbols $f \in F$ with arity $ar(f) \geq 0$, is a standard predicate logic extended by a *finite* number of universal modal operators $\Box_i, i \geq 1$. In this case we do not require that these universal modal operators are normal (that is, monotonic and multiplicative) modal operators as in the standard setting for modal logics but we require that they have the same standard Kripke semantics. In the standard Kripke semantics, each modal operator $\Box_i$ is defined by an accessibility binary relation $\mathcal{R}_i \subseteq \mathcal{W} \times \mathcal{W}$ for a given set of possible worlds $\mathcal{W}$.
We define the set of terms of this predicate modal logic as follows (Var denotes the set of variables and S the set of constants): all variables $x \in Var$ and constants $d \in S$ are terms; if $f \in F$ is a functional symbol of arity $k = ar(f)$ and $t_1, .., t_k$ are terms then $f(t_1, .., t_k)$ is a term. We denote the set of all ground (without variables) terms by $\mathcal{T}_0$.
An atomic formula (atom) for a predicate symbol $r \in P$ with arity $k = ar(r)$ is an expression $r(t_1, ..., t_k)$, where $t_i, i = 1, ..., k$ are terms. Herbrand base H is a set of all ground atoms (atoms without variables). More complex formulae, for a predicate multi-modal logic, are obtained as a free algebra obtained from the set of all atoms and usual set of classic 2-valued binary logic connectives in $\{\wedge, \vee, \Rightarrow\}$ for conjunction, disjunction and implication respectively (negation of a formula ϕ, denoted by $\neg\phi$ is expressed by $\phi \Rightarrow 0$, where 0 is used for an inconsistent formula (has value 0 constantly for every valuation), and a number of unary universal modal operators $\Box_i$. We define $\mathcal{N} = \{1, ..., n\}$, where n is the maximal arity of symbols in the finite set $P \bigcup F$.

Definition 1. *We denote a multi-modal Kripke model by* $\mathcal{M} = (\mathcal{W}, \{\mathcal{R}_i \mid 1 \leq i \leq k\}, S, V)$*, with finite* $k \geq 1$ *modal operators with a set of possible worlds* $\mathcal{W}$*, the accessibility relations* $\mathcal{R}_i \subseteq \mathcal{W} \times \mathcal{W}$*, non empty set of individuals* S*, and a function* $V : \mathcal{W} \times (P \bigcup F) \rightarrow \bigcup_{n \in \mathcal{N}} (\mathbf{2} \bigcup S)^{S^n}$*, such that for any world* $w \in \mathcal{W}$*,*
1. For any functional letter $f \in F$*,* $V(w, f) : S^{ar(f)} \rightarrow S$ *is a function (interpretation of* f *in* w*).*
2. For any predicate letter $r \in P$*, the function* $V(w, r) : S^{ar(r)} \rightarrow \mathbf{2}$ *defines the extension of* r *in a world* w*,* $\|r\| = \{\mathbf{d} =< d_1, ..., d_k >\in S^k \mid k = ar(r), V(w, r)(\mathbf{d}) = 1\}$.

We denote the fact that a formula φ is satisfied in a world $w \in \mathcal{W}$ for a given assignment $g : Var \rightarrow S$ by $\mathcal{M} \models_{w,g} \varphi$. For example, a given atom $r(x_1, ..., x_k)$ is satisfied in w by assignment g, i.e., $\mathcal{M} \models_{w,g} r(x_1, ..., x_k)$ iff $V(w, r)(g(x_1), ..., g(x_k)) = 1$.
The Kripke semantics is extended to all formuale as follows:
$\mathcal{M} \models_{w,g} \varphi \wedge \phi$ iff $\mathcal{M} \models_{w,g} \varphi$ and $\mathcal{M} \models_{w,g} \phi$,
$\mathcal{M} \models_{w,g} \varphi \vee \phi$ iff $\mathcal{M} \models_{w,g} \varphi$ or $\mathcal{M} \models_{w,g} \phi$,
$\mathcal{M} \models_{w,g} \varphi \Rightarrow \phi$ iff $\mathcal{M} \models_{w,g} \varphi$ implies $\mathcal{M} \models_{w,g} \phi$,
$\mathcal{M} \models_{w,g} \Box_i\varphi$ iff $\forall w'((w, w') \in \mathcal{R}_i$ implies $\mathcal{M} \models_{w',g} \varphi)$.
The existential modal operator $\Diamond_i$ can be defined as a derived operator by taking $\neg\Box_i\neg$.
A formula φ is said to be *true in a model* $\mathcal{M}$ if for each assignment function g and possible world w, $\mathcal{M} \models_{w,g} \varphi$. A formula is said to be *valid* if it is true in each model.

We denote the set of all worlds where the ground formula ϕ/g (obtained from ϕ and an assignment g) is satisfied by $\|\phi/g\| = \{w \mid \mathcal{M}\} \models_{w,g} \phi\}$.
Remark: in this paper we use the notation $[x]$ (for any truth value $x \in X$) for universal modal operators, instead of standard notation $\Box_i$.

1.3 Introduction to binary sequents and bivaluations

Sequent calculus has been developed by Gentzen [2], inspired on ideas of Paul Hertz [3]. Given a propositional logic language $\mathcal{L}_A$ (a set of logic formulae), a sequent is a consequence pair of formulae, $s = (\phi; \psi) \in \mathcal{L}_A \times \mathcal{L}_A$, denoted also by $\phi \vdash \psi$.
A Gentzen system, denoted by a pair $\mathcal{G} = \langle \mathbb{L}, \Vdash \rangle$, where $\Vdash$ is a consequence relation on a set of sequents in $\mathbb{L} \subseteq \mathcal{L}_A \times \mathcal{L}_A$, is said to be *normal* if it satisfies the following conditions: for any sequent $s = \phi \vdash \psi \in \mathbb{L}$ and a set of sequents $\Gamma = \{s_i = \phi_i \vdash \psi_i \in \mathbb{L} \mid i \in I\}$,
1. reflexivity: if $s \in \Gamma$ then $\Gamma \Vdash s$
2. transitivity: if $\Gamma \Vdash s$ and for every $s' \in \Gamma$, $\Theta \Vdash s'$, then $\Theta \Vdash s$
3. finiteness: if $\Gamma \Vdash s$ then there is finite $\Theta \subseteq \Gamma$ such that $\Theta \Vdash s$.
4. for any homomorphism σ from $\mathbb{L}$ into itself (i.e., a substitution), if $\Gamma \Vdash s$ then $\sigma[\Gamma] \Vdash \sigma(s)$, i.e., $\{\sigma(\phi_i) \vdash \sigma(\psi_i) \mid i \in I\} \Vdash (\sigma(\phi) \vdash \sigma(\psi))$.
Notice that from (1) and (2) we obtain this monotonic property:
5. if $\Gamma \Vdash s$ and $\Gamma \subseteq \Theta$, then $\Theta \Vdash s$.
We denote the Tarskian closure operator by $C : \mathcal{P}(\mathbb{L}) \rightarrow \mathcal{P}(\mathbb{L})$, such that $C(\Gamma) =_{def} \{s \in \mathbb{L} \mid \Gamma \Vdash s\}$, with the properties: $\Gamma \subseteq C(\Gamma)$ (from reflexivity (1)); it is monotonic, i.e., $\Gamma \subseteq \Gamma_1$ implies $C(\Gamma) \subseteq C(\Gamma_1)$ (from (5)), and an involution $C(C(\Gamma)) = C(\Gamma)$ as well. Thus, we obtain
6. $\Gamma \Vdash s$ iff $s \in C(\Gamma)$.
Any sequent theory $\Gamma \subseteq \mathbb{L}$ is said to be a *closed* theory iff $\Gamma = C(\Gamma)$. This closure property corresponds to the fact that $\Gamma \Vdash s$ iff $s \in \Gamma$.
Each sequent theory Γ can be considered as a bivaluation (a characteristic function) $\beta : \mathbb{L} \rightarrow \mathbf{2}$ such that for any sequent $s \in \mathbb{L}$, $\beta(s) = 1$ iff $s \in \Gamma$.

2 Reduction of finite many-valued logic into 2-valued multi-modal logic

Let $\mathcal{L}_P$ be a predicate logic language obtained as a free algebra from connectives in Σ of an algebra with a set X of truth values (for example the many-valued conjunction, disjunction and implication $\{\wedge_m, \vee_m, \Rightarrow_m\} \subseteq \Sigma$ are binary operators, negation $\neg_m \in \Sigma$ and other modal operators are unary operators, while each $x \in X \subseteq \Sigma$ is a constant (nullary operator)), a set P of predicate symbols denoted by $p, r, q, ..$ with a given arity (in case when the arity of all symbols in P is a zero we obtain that P is the set of propositional variables (letters), so that $\mathcal{L}_P$ is a propositional logic), and a set F of functional symbols (with a given arity) denoted by f, g, h.
We define the set of terms of this logic as follows: all variables $\nu_i \in Var$, $i = 1, 2, ...$ and constants $d \in S$ are terms; if $f \in F$ is a functional symbol of arity n and $t_1, .., t_n$ are terms, then $f(t_1, .., t_n)$ is a term. The ground term is a term without variables. We

denote the set of all terms by $\mathcal{T}_S$.
The set of atoms is defined as $\mathcal{A}_S = \{p(c_1, .., c_n) \mid p \in P,\ n = arity(p)$ and $c_i \in \mathcal{T}_S\}$. The set of all ground atoms (without variables), $H = \{p(c_1/g, .., c_n/g) \mid p \in P,\ n = arity(p)\ ,\ c_i \in \mathcal{T}_S$ and $g : Var \to S\}$, is a Herbrand base (here c_i/g denotes a ground atom obtained from a term c_i by an assignment g).
Any atom is a logic formulae. The combination of logic formulae by logical connectives in Σ is another logic formula.
We denote the subset of the logic language $\mathcal{L}_P$ composed of only *ground* formulae by $\mathcal{L}$. In this case we can represent each ground atom by a particular propositional letter, so that this logic language $\mathcal{L}$ is equivalent to the propositional logic language where the ground atoms in $H \subseteq \mathcal{L}$ are replaced by propositional letters $A, B, ...$
We use the letters ϕ, ψ for formulae of $\mathcal{L}$.
We define a (many-valued) *valuation* v as a mapping $v : H \to X$, which is uniquely extended in standard way to the homomorphism $v : \mathcal{L} \to X$ (for example, for any $A, B \in H$, $v(A \odot B) = v(A) \odot v(B), \odot \in \{\wedge_m, \vee_m, \Rightarrow_m\}$ and $v(\neg_m A) = \neg_m v(A)$, where $\wedge_m, \vee_m, \Rightarrow_m, \neg_m$ are the many-valued conjunction, disjunction, implication and negation respectively). The set of all *many-valued valuations* is a strict subset $\mathbb{V}_m$ of the functional space $X^{\mathcal{L}}$ that satisfies the homomorphic conditions above.
Based on this propositional many-valued logic language $\mathcal{L}$, we are able to define the following multi-modal 2-valued algebraic logic language $\mathcal{L}_M^*$, by introducing the modal non-standard (non monotonic) algebraic truth-functional operators $[x] : Y \to \mathbf{2}$, where $Y = X \bigcup \mathbf{2}$ such that for any $x, y \in Y$, $[x](y) = 1$ iff $x = y$ [18]. The intersection of X and $\mathbf{2}$ can be non empty as well.
The set of truth values X is a finite set, so that the number of these algebraic modal operators $n = |Y| \geq 2$ is finite as well ($|Y|$ is the cardinality of the set Y).

Definition 2. SYNTAX: *Let $\mathcal{L}_P$ be a predicate many-valued logic language with a set of truth values X and $(\mathbf{2}, \wedge, \vee)$ be the complete distributive two-valued lattice. The multi-modal 2-valued logic language $\mathcal{L}_M^*$ is the set of all modal formulae (we will use letters Φ, Ψ.. for the formulae of $\mathcal{L}_M^*$) defined as follows:*
1. $\mathbf{2} \subseteq \mathcal{L}_M^*$.
2. $[x]\phi \in \mathcal{L}_M^*$, *for any* $x \in X, \phi \in \mathcal{L}_P$.
3. $[x]\Phi \in \mathcal{L}_M^*$, *for any* $x \in \mathbf{2}, \Phi \in \mathcal{L}_M^*$.
4. $\Phi, \Psi \in \mathcal{L}_M^*$ *implies* $\Phi \wedge \Psi, \Phi \vee \Psi \in \mathcal{L}_M^*$.
We denote the sublanguage of $\mathcal{L}_M^*$ without variables by $\mathcal{L}_M$(ground atoms in $\mathcal{L}_M$ are considered as propositional letters).

The constants $0, 1$ correspond to the tautology and contradiction proposition respectively, and they can be considered as nullary operators in $\mathcal{L}_M$.
We can use this 2-valued multi-modal logic language $\mathcal{L}_M$ in order to define the sequents as elements of the Cartesian product $\mathcal{L}_M \times \mathcal{L}_M$, i.e., each sequent s is denoted by $\Phi \vdash \Psi$, where $\Phi, \Psi \in \mathcal{L}_M$.

Definition 3. SEMANTICS: *For any many-valued valuation $v \in \mathbb{V}_m$, $v : \mathcal{L} \to X$, we define the 'modal valuation' $\alpha : \mathcal{L}_M \to \mathbf{2}$ as follows:*
1. $\alpha(0) = 0,\ \ \alpha(1) = 1.$
2. $\alpha([x]\phi) = 1$ *iff* $x = v(\phi)$, *for any* $x \in X$, $\phi \in \mathcal{L}$.

3. $\alpha([x]\Phi) = 1$ *iff* $x = \alpha(\Phi)$*, for any* $x \in \mathbf{2}$*,* $\Phi \in \mathcal{L}_M$*.*
4. $\alpha(\Phi \wedge \Psi) = \alpha(\Phi) \wedge \alpha(\Psi)$*,* $\alpha(\Phi \vee \Psi) = \alpha(\Phi) \vee \alpha(\Psi)$*, for any* $\Phi, \Psi \in \mathcal{L}_M$*.*
This transformation from many-valued valuations into modal valuations can be expressed by the mapping $\mathfrak{F} : \mathbb{V}_m \to \mathcal{V}$*, where* $\mathcal{V} \subset \mathbf{2}^{\mathcal{L}_M}$ *denotes the set of all modal valuations.*

It is easy to verify that the mapping $\mathfrak{F}$ is a bijection, with its inverse $\mathfrak{F}^{-1}$ defined as follows: for any modal valuation $\alpha \in \mathcal{V}$, given in Definition 3, the many-valued valuation $v = \mathfrak{F}^{-1}(\alpha) : \mathcal{L} \to X$ is defined for any $\phi \in \mathcal{L}$, by $v(\phi) = x \in X$ iff $\alpha([x]\phi) = 1$.
A many-valued valuation $v : \mathcal{L} \to X$, $v \in \mathbb{V}_m$ satisfies a 2-valued multi-modal formula $\Phi \in \mathcal{L}_M$ iff $\mathfrak{F}(v)(\Phi) = 1$.
Given two formulae $\Phi, \Psi \in \mathcal{L}_M$, the sequent $\Phi \vdash \Psi$ is satisfied by v if $\mathfrak{F}(v)(\Phi) \leq \mathfrak{F}(v)(\Psi)$.
A sequent $\Phi \vdash \Psi$ is an axiom if it is satisfied by every valuation $v \in \mathbb{V}_m \subset X^{\mathcal{L}}$.
From this definition of satisfaction for sequents, we obtain the reflexivity (axiom) $\Phi \vdash \Phi$ and transitivity (cut) inference rule, i.e., from $\Phi \vdash \Psi$ and $\Psi \vdash \Upsilon$ we deduce $\Phi \vdash \Upsilon$.
Let us define the set of 2-valued multi-modal literals (or modal atoms) as
$P_{mm} = \{[x_1]...[x_k]A \in \mathcal{L}_M \mid k \geq 1 \text{ and } A \in H\}$.
For example, if $v(\phi) = x$ then $\mathfrak{F}(v)([1][x]\phi) = 1$, while if $v(\phi) \neq x$ then $\mathfrak{F}(v)([0][x]\phi) = 1$. Notice that the number of nested modal operators can be reduced from the fact that $[0][0]$ and $[1][1]$ are identities for the formulae in $\mathcal{L}_M$. For example, for $[x]\phi \in \mathcal{L}_M$, we have $[0][1][1][0][x]\phi \equiv [0][0][x]\phi \equiv [x]\phi$, where $\equiv$ is a standard logic equivalence.
Then, given a formula $\phi \in \mathcal{L}$, the modal formula $[x]\phi \in \mathcal{L}_M$ can be naturally reduced to an equivalent formula, denoted by $\widehat{[x]\phi}$, where the modal operators $[x]$ are applied only to ground atoms (considered as propositional letters) in H. Moreover, for any formula $\Phi \in \mathcal{L}_M$, there is an equivalent formula $\widehat{\Phi}$ composed by logical connectives $\wedge$, $\vee$, and by multi-modal literals in P_{mm}. A canonical formula can be obtained by the following reduction:

Definition 4. CANONICAL REDUCTION: *Let us define the following reduction rules:*
1. For any unary operator $\sim \in \Sigma$*,* $\phi \in \mathcal{L}$*, and a value* $x \in X$*,*
$[x](\sim \phi) \quad \mapsto \quad \bigvee_{y \in X.\ x = \sim y}[y]\phi$*,*
2. For any binary operator $\odot \in \Sigma$*,* $\phi, \psi \in \mathcal{L}$*, and a value* $x \in X$*,*
$[x](\phi \odot \psi) \quad \mapsto \quad \bigvee_{y,z \in X.\ x = y \odot z}([y]\phi \wedge [z]\psi)$*.*
3. For any binary operator $\odot \in \{\wedge, \vee\}$*,* $\Phi, \Psi \in \mathcal{L}_M$*, and a value* $x \in \mathbf{2}$*,*
$[x](\Phi \odot \Psi) \quad \mapsto \quad \bigvee_{y,z \in \mathbf{2}.\ x = y \odot z}([y]\Phi \wedge [z]\Psi)$*.*
We denote the canonic formula obtained by applying recursively these reduction rules to the formula $[x]\Phi$ *by* $\widehat{[x]\Phi}$*.*

Proposition 1 *The normal reductions in Definition 4 are truth-preserving, that is, for any* $x \in X$ *and* $\phi \in \mathcal{L}$ *we have that* $[x]\phi$ *is logically equivalent to* $\widehat{[x]\phi}$*.*
Analogously, for any $x \in \mathbf{2}$ *and* $\Phi \in \mathcal{L}_M$*, we have that* $[x]\Phi$ *is logically equivalent to* $\widehat{[x]\Phi}$*.*

Proof: Let us show that the steps of canonical reduction in Definition 4 are truth-preserving:
1. The first case of reduction: let us suppose that $[x](\sim \phi)$ is true but $\bigvee_{y \in X. x = \sim y}[y]\phi$

is false. From the truth of $[x](\sim \phi)$, we obtain that $x = v(\sim \phi)$: let $z = v(\phi)$ and, consequently, $[z]\phi$ is true then, from the truth-functional connective $\sim$, we conclude that $x =\sim z$ and from the fact that $[z]\phi$ is true, we conclude that $\bigvee_{y \in X. x = \sim y}[y]\phi$ must be true, which is a contradiction. Consequently $[x](\sim \phi)$ implies $\bigvee_{y \in X. x = \sim y}[y]\phi$.
Vice versa, if $\bigvee_{y \in X. x = \sim y}[y]\phi$ is true then there exists z such that $[z]\phi$ is true, with $x =\sim z$, i.e., $x = v(\sim \phi)$ so that $[x](\sim \phi)$ is true.
Consequently, $[x](\sim \phi)$ iff $\bigvee_{y \in X. x = \sim y}[y]\phi$.
2. Analogously, for the second case (and 3rd as well) we obtain that
$[x](\phi \odot \psi)$ iff $\bigvee_{y \in X. x = y \odot z}([y]\phi \wedge [z]\psi)$.
From the fact that both steps are also truth-preserving, we deduce that any consecutive execution of them is truth-preserving and consequently $[x]\phi$ iff $\widehat{[x]\phi}$.
□
The following proposition shows that the result of the canonical reduction of a formula $\Phi \in \mathcal{L}_M$ is a disjunction of modal conjunctions, which in the case of the formulae without nested modal operators is a simple disjunctive modal formula.

Proposition 2 *Any 2-valued logic formulae $\Phi \in \mathcal{L}_M$ is logically equivalent to disjunctive modal formula $\bigvee_{1 \le i \le m}(\bigwedge_{1 \le j \le m_i}([y_{ij1}]...[y_{ijk_{ij}}])A_{ij})$, where for all $1 \le i \le m$, and $1 \le j \le m_i$, we have that $1 \le k_{ij}$, $[y_{ijk_{ij}}] \in X$, and $A_{ij} \in H$.*
In the case when we have no nested modal operators then $k_{ij} = 1$ for all i, j. Consequently, $\Phi \in \mathcal{L}_M$ is logically equivalent to a disjunctive modal formula $\bigvee_{1 \le i \le m}[x_i]\phi_i$, where for all $1 \le i \le m$, $x_i \in X$, $\phi_i \in \mathcal{L}$.

Proof: From Proposition 1 and from the canonical reduction in Definition 4, it is easy to see that each logically equivalent reduction moves a modal operator towards propositional letters in H, by the introduction of conjunction and disjunction logic operators only. Thus, when this reduction is completely realized, we obtain a positive propositional logic formula with modal propositions in P_{mm} and logic operators $\wedge$ and $\vee$. It is well known that such a positive propositional formula can be equivalently (but not uniquely) represented as a disjunction of conjunctions of modal propositions $([y_{ij1}]...[y_{ijk_{ij}}])A_{ij} \in P_{mm}$.
Let us now consider a formula $\Phi \in \mathcal{L}_M$ without nested modal operators. Then $k_{ij} = 1$ for all i, j and, from the result above, a formula $\Phi \in \mathcal{L}_M$ can be equivalently (but not uniquely) represented as a disjunction of conjunctive forms $\bigvee_{1 \le i \le m}(\bigwedge_{1 \le j \le m_i}[y_{ij1}]A_{ij})$.
But we have that $\bigwedge_{1 \le j \le m_i}[y_{ij1}]A_{ij} = [x_i]\phi_i$, where for any binary operator $\odot \in \Sigma$, $x_i = y_{i11} \odot ... \odot y_{im_i1} \in X$, and $\phi_i = A_{i1} \odot ... \odot A_{im_i} \in \mathcal{L}$. It holds because $\alpha([x_i]\phi_i) = 1$ iff $x_i = v(\phi_i) = v(A_{i1} \odot ... \odot A_{im_i}) = v(A_{i1}) \odot ... \odot v(A_{im_i}) = y_{i11} \odot ... \odot y_{im_i1}$.
That is, Φ iff $\bigvee_{1 \le i \le m}[x_i]\phi_i$.
□
With this normal reduction, by using truth-value tables of many-valued logical connectives, we introduced the structural compositionality and truth preserving for the 2-valued modal encapsulation of a many-valued logic $\mathcal{L}$ as well. In fact, the following property is valid:

Proposition 3 *Given a many-valued valuation $v : \mathcal{L} \to X$ and a formula $\phi \in \mathcal{L}$, the normal reduct formula $\widehat{[x]\phi} \in \mathcal{L}_M$ is satisfied by v iff $x = v(\phi)$.*

Proof: By structural induction on ϕ: let $\alpha = \mathfrak{F}(v)$ then we have the following cases:
1. when $\phi = A \in H$:
If $\widehat{[x]A}$ is satisfied then $\alpha(\widehat{[x]\phi}) = \alpha([x]A) = 1$ (from Definition 3 and Proposition 1), thus $x = v(A) = v(\phi)$.
2. when $\phi = \sim \psi$, where $\sim$ is an unary operator in Σ, with $y = v(\psi)$ and $x = \sim y$:
Suppose, by structural induction, that $\widehat{[y]\psi}$ is satisfied, i.e., $y = v(\psi)$. As result, $[y]\psi$ is true (from Definition 3). Thus, $\bigvee_{y \in X. x = \sim y}[y]\psi$ is true and by the truth-preservation (from the reduction 1 in Definition 4) we obtain that $[x](\sim \psi)$ is true, so that, $[x]\phi$ is true, i.e., (by Definition 3) $x = v(\phi)$, and (from Proposition 1) $\widehat{[x]\phi}$ is true. With $x = \sim y = \sim (v(\psi)) =$ (from the homomorphism of v) $= v(\sim \psi) = v(\phi)$.
3. when $\phi = \psi \odot \varphi$, where $\odot$ is a binary operator in Σ, with $y = v(\psi)$, $z = v(\varphi)$ and $x = y \odot z$:
By structural induction hypothesis, $\widehat{[y]\psi}$ and $\widehat{[z]\varphi}$ are satisfied, so that from Proposition 1 and Definition 3, we have that $1 = \alpha(\widehat{[y]\psi}) = \alpha([y]\psi)$ and $1 = \alpha(\widehat{[z]\varphi}) = \alpha([z]\varphi)$. Consequently, $1 = \alpha([y]\psi) \wedge \alpha([z]\varphi) =$ (from the homomorphism 2 in Definition 3) $= \alpha([y]\psi \wedge [z]\varphi) = \alpha(\bigvee_{y \in X. x = y \odot z}([y]\psi \wedge [z]\varphi)) =$ (from the reduction 2 in Definition 4) $= \alpha([x](\psi \odot \varphi)) = \alpha([x]\phi) =$ (by Proposition 1) $= \alpha(\widehat{[x]\phi})$. Thus, $\widehat{[x]\phi}$ is satisfied by v and $x = y \odot z = v(\psi) \odot v(\varphi) =$ (from the homomorphism of v) $= v(\psi \odot \varphi) = v(\phi)$.
□
Thus, as a consequence, for any $\phi \in \mathcal{L}$ and a many-valued valuation $v \in \mathbb{V}_m$, we have that $\mathfrak{F}(v)(\widehat{[x]\phi}) = 1$ iff $x = v(\phi)$.

3 Many-valued truth and model theoretic semantics

The Gentzen-like system $\mathcal{G}$ of the 2-valued propositional logic $\mathcal{L}_M$ (where the set of propositional letters corresponds to the set $P_{mm} = \{[x_1]...[x_k]A \in \mathcal{L}_M \mid k \geq 1$ and $A \in H\}$ is a 2-valued distributive logic (DLL in [19]), that is, $\mathbf{2} \subseteq \mathcal{L}_M$, extended by the set of sequent axioms, defined for each many-valued logic connective in Σ of the original many-valued logic $\mathcal{L}$. It contains the following axioms (sequents) and rules:

(AXIOMS) The Gentzen-like system $\mathcal{G} = \langle \mathbb{L}, \Vdash \rangle$ contains the following sequents in $\mathbb{L}$ for any $\Phi, \Psi, \Upsilon \in \mathcal{L}_M$:
1. $\Phi \vdash \Phi$ (reflexive)
2. $\Phi \vdash 1$, $0 \vdash \Phi$ (top/bottom axioms)
3. $\Phi \wedge \Psi \vdash \Phi$, $\Phi \wedge \Psi \vdash \Psi$ (product projections: axioms for meet)
4. $\Phi \vdash \Phi \vee \Psi$, $\Phi \vdash \Psi \vee \Phi$ (coproduct injections: axioms for join)
5. $\Phi \wedge (\Psi \vee \Upsilon) \vdash (\Phi \wedge \Psi) \vee (\Phi \wedge \Upsilon)$ (distributivity axiom)
6. The set of Introduction axioms for many-valued connectives:
6.1 $\bigvee_{y,z \in \mathbf{2}.\ x_1 = y \odot z}([y]\Phi \wedge [z]\Psi) \ \vdash \ [x_1](\Phi \odot \Psi)$ for any binary operator $\odot \in \{\wedge, \vee\}$.
6.2 $\bigvee_{y \in X. x = \sim y}[y]\phi \ \vdash \ [x](\sim \phi)$ for any unary operator $\sim \in \Sigma$ and $\phi \in \mathcal{L}$.

6.3 $\bigvee_{y,z \in X. x=y \odot z}([y]\phi \wedge [z]\psi) \ \ \vdash \ \ [x](\phi \odot \psi)$ for any binary operator $\odot \in \Sigma$ and $\phi, \psi \in \mathcal{L}$.

7. The set of elimination axioms for many-valued connectives:

7.1 $[x_1](\Phi \odot \Psi) \ \vdash \ \bigvee_{y,z \in \mathbf{2}.\ x_1 = y \odot z}([y]\Phi \wedge [z]\Psi)$ for any binary operator $\odot \in \{\wedge, \vee\}$.

7.2 $[x](\sim \phi) \ \vdash \ \bigvee_{y \in X. x = \sim y}[y]\phi$ for any unary operator $\sim \in \Sigma$ and $\phi \in \mathcal{L}$.

7.3 $[x](\phi \odot \psi) \ \ \vdash \ \ \bigvee_{y,z \in X. x=y \odot z}([y]\phi \wedge [z]\psi)$ for any binary operator $\odot \in \Sigma$ and $\phi, \psi \in \mathcal{L}$.

(INFERENCE RULES) $\mathcal{G}$ is closed under the following inference rules:

1. $\frac{\Phi \vdash \Psi, \ \Psi \vdash \Upsilon}{\Phi \vdash \Upsilon}$ (cut/ transitivity rule)
2. $\frac{\Phi \vdash \Psi, \ \Phi \vdash \Upsilon}{\Phi \vdash \Psi \wedge \Upsilon}$, $\frac{\Phi \vdash \Psi, \ \Upsilon \vdash \Psi}{\Phi \vee \Upsilon \vdash \Psi}$ (lower/upper lattice bound rules)
3. $\frac{\Phi \vdash \Psi}{\sigma(\Phi) \vdash \sigma(\Psi)}$ (substitution rule: σ is substitution (γ/p)).

□

The axioms from 1 to 5 and the rules 1 and 2 are taken from [19] for the DLL and it was shown that this sequent based Genzen-like system is sound and complete. The new axioms 6 and 7 correspond to the canonical (equivalent) reductions in Definition 4. The set of sequents that define the poset of the classic 2-valued lattice of truth values $(\mathbf{2}, \leq)$ is a consequence of the top/bottom axioms: for any two $x, y \in \mathbf{2}$, if $x \leq y$ then $x \vdash y \in \mathcal{G}$.

Thus, for many-valued logics, we obtain a *normal* modal Gentzen-like deductive system, where each sequent is a valid truth-preserving consequence-pair defined by the poset of the complete lattice $(\mathbf{2}, \leq)$ of classic truth values (which are also the constants of this positive propositional language $\mathcal{L}_M$), so that each occurrence of the symbol $\vdash$ can be substituted by the partial order $\leq$ of this complete lattice $(\mathbf{2}, \leq)$.

Example 1: Let us consider the Godel's 3-valued logic $X = \{0, \frac{1}{2}, 1\}$, and its 3-valued implication logic connective $\Rightarrow$ given by the following truth-table:

$\Rightarrow$	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	0	1	1
1	0	$\frac{1}{2}$	1

One of the possible m-sequents for introduction *rules* for this connective, taken from [12], (each rule corresponds to the conjunction of disjunctive forms, where each disjunctive form is one m-sequent in the premise), is

$$\frac{\langle \Gamma | \triangle, \phi | \Pi, \phi \rangle \ \ \langle \Gamma_1, \psi | \triangle_1 | \Pi_1 \rangle}{\langle \Gamma, \Gamma_1, \phi \Rightarrow \psi | \triangle, \triangle_1 | \Pi, \Pi_1 \rangle} \ \Rightarrow: 0, \quad \frac{\langle \Gamma | \triangle | \Pi, \phi \rangle \ \ \langle \Gamma_1 | \triangle_1, \psi | \Pi_1 \rangle}{\langle \Gamma, \Gamma_1, | \triangle, \triangle_1, \phi \Rightarrow \psi | \Pi, \Pi_1 \rangle} \ \Rightarrow: \frac{1}{2}$$
$$\frac{\langle \Gamma, \phi | \triangle, \phi | \Pi, \psi \rangle \ \ \langle \Gamma_1, \phi | \triangle_1, \psi | \Pi_1, \psi \rangle}{\langle \Gamma, \Gamma_1, | \triangle, \triangle_1 | \Pi, \Pi_1, \phi \Rightarrow \psi \rangle} \ \Rightarrow: 1$$

but in our approach, we obtain the *unique* set of binary sequent introduction *axioms* as follows:

$([\frac{1}{2}]\phi \wedge [0]\psi) \vee ([1]\phi \wedge [0]\psi) \ \ \vdash \ \ [0](\phi \Rightarrow \psi)$
$[1]\phi \wedge [\frac{1}{2}]\psi \ \ \vdash \ \ [\frac{1}{2}](\phi \Rightarrow \psi)$
$\bigvee_{x,y \in X \ \& \ (x,y) \notin S} ([x]\phi \wedge [y]\psi) \ \ \vdash \ \ [1](\phi \Rightarrow \psi)$

where $S = \{(\frac{1}{2}, 0), (1, 0), (1, \frac{1}{2})\}$,
and elimination axioms:
$[0](\phi \Rightarrow \psi) \;\vdash\; ([\frac{1}{2}]\phi \wedge [0]\psi) \vee ([1]\phi \wedge [0]\psi)$
$[\frac{1}{2}](\phi \Rightarrow \psi) \;\vdash\; [1]\phi \wedge [\frac{1}{2}]\psi$
$[1](\phi \Rightarrow \psi) \;\vdash\; \bigvee_{x,y \in X \;\&\; (x,y) \notin S} ([x]\phi \wedge [y]\psi)$.
□

Definition 5. *For any two formulae $\Phi, \Psi \in \mathcal{L}_M$ when the sequent $\Phi \vdash \Psi$ is satisfied by a 2-valued modal valuation $\alpha : \mathcal{L}_M \to \mathbf{2}$ from Definition 3 (that is, when $\alpha(\Phi) \leq \alpha(\Psi)$ as in standard 2-valued logics), we say that it is satisfied by the many-valued valuation $v = \mathfrak{F}^{-1}(\alpha) : \mathcal{L} \to X$.*
This sequent is a tautology if it is satisfied by all modal valuations $\alpha \in \mathcal{V}$, i.e., when $\forall v \in \mathbb{V}_m.(\mathfrak{F}(v)(\Phi) \leq \mathfrak{F}(v)(\Psi))$.
For a normal Gentzen-like sequent system $\mathcal{G} = \langle \mathbb{L}, \Vdash \rangle$ of a many-valued logic language $\mathcal{L}$, with the set of sequents $\mathbb{L} \subseteq \mathcal{L}_M \times \mathcal{L}_M$, we tell that a many-valued valuation v is its `model` *if it satisfies all sequents in $\mathcal{G}$.*
The set of all models of a given set of sequents (theory) Γ is:
$Mod_\Gamma = \{v \in \mathbb{V}_m \mid \forall(\Phi \vdash \Psi) \in \Gamma(\mathfrak{F}(v)(\Phi) \leq \mathfrak{F}(v)(\Psi))\} \;\subseteq\; \mathbb{V}_m \subset X^{\mathcal{L}}$.

Proposition 4 SEQUENT'S BIVALUATIONS AND SOUNDNESS: *Let us define the mapping $\mathfrak{B} : \mathbb{V}_m \to \mathbf{2}^{\mathcal{L}_M \times \mathcal{L}_M}$ from valuations into sequent bivaluations such that for any valuation $v \in \mathbb{V}_m$, we obtain the sequent bivaluation $\beta = \mathfrak{B}(v) = eq \circ < \pi_1, \wedge > \circ(\mathfrak{F}(v) \times \mathfrak{F}(v)) : \mathcal{L}_M \times \mathcal{L}_M \to \mathbf{2}$, where π_1 is the first projection, $\circ$ is the functional composition and $eq : \mathbf{2} \times \mathbf{2} \to \mathbf{2}$ is the equality mapping such that $eq(x, y) = 1$ iff $x = y$.*
Then, a sequent $s = (\Phi \vdash \Psi)$ is satisfied by v iff $\beta(s) = \mathfrak{B}(v)(s) = 1$.
All axioms of the Gentzen like sequent system $\mathcal{G}$, of a many-valued logic language $\mathcal{L}$ based on a set X of truth values, are tautologies, and all its rules are sound for model satisfiability and preserve the tautologies.

Proof: From the definition of a bivaluation β, we have that
$\beta(\Phi \vdash \Psi) = \beta(\Phi; \Psi) = eq \circ < \pi_1, \wedge > \circ(\mathfrak{F}(v) \times \mathfrak{F}(v))(\Phi; \Psi)$
$= eq \circ < \pi_1, \wedge > (\mathfrak{F}(v)(\Phi) \times \mathfrak{F}(v)(\Psi))$
$= \; eq < \pi_1(\mathfrak{F}(v)(\Phi), \mathfrak{F}(v)(\Psi)), \; \wedge(\mathfrak{F}(v)(\Phi), \mathfrak{F}(v)(\Psi)) >$
$= \; eq(\mathfrak{F}(v)(\Phi), \; \mathfrak{F}(v)(\Phi) \wedge \mathfrak{F}(v)(\Psi))$.
Thus $\beta(\Phi \vdash \Psi) = 1$ iff $\mathfrak{F}(v)(\Phi) \leq \mathfrak{F}(v)(\Psi)$, i.e., when this sequent is satisfied by v.
It is straightforward to check that all axioms in $\mathcal{G}$ are tautologies (all constant sequents specify the poset of the complete lattice $(\mathbf{2}, \leq)$ of classic 2-valued logic, thus they are tautologies). It is straightforward to check that all rules preserve the tautologies. Moreover, if all premises of a given rule in $\mathcal{G}$ are satisfied by a given many-valued valuation $v : \mathcal{L} \to X$ then also the deduced sequent of this rule is satisfied by the same valuation, i.e., the rules are sound for the model satisfiability.
□
It is easy to verify that this entailment is equal to the classic propositional entailment.
Remark: It is easy to observe that each sequent is, from the logic point of view, a *2-valued object* so that all inference rules are embedded into the classic 2-valued framework, i.e., given a bivaluation $\beta = \mathfrak{B}(v) : \mathcal{L}_M \times \mathcal{L}_M \to \mathbf{2}$, we have that a sequent

$s = \Phi \vdash \Psi$ is satisfied, $\beta(s) = 1$ iff $\mathfrak{F}(v)(\Phi) \leq \mathfrak{F}(v)(\Psi)$, so that we have a direct relationship between sequent bivaluations β and many-valued valuations v.
This sequent feature, which is only an alternative formulation for the 2-valued classic logic, is *fundamental* in the framework of many-valued logics where the semantics for the entailment based on the algebraic matrices (X, D) is often subjective and arbitrary. Let us consider, for example, the fuzzy logic with the uniquely *fixed semantics* for all logical connectives, where the subset of designated elements $D \subseteq X$ is an arbitrary-subjective choice between the *infinite* number of closed intervals $[a, 1]$ also for very restricted interval. For example, $0.83 \leq a \leq 0.831$. It does not happen in the classic 2-valued logics where a different logic semantics (entailment) is obtained by adopting only different semantics for some of its logical connectives, usually for negation operator. This property of the classic 2-valued logic can be propagated to many-valued logics by adopting the principle of classic truth-invariance for the entailment. In that case, for fixed semantics of all logical connectives of a given language, we obtain a unique logic. The definition of the 2-valued entailment in the sequent system $\mathcal{G}$, given in Definition 5, can replace the current entailment based on algebraic matrices (X, D) where $D \subseteq X$ is a subset of designated elements. Thus, we are now able to introduce the many-valued valuation-based (i.e., model-theoretic) semantics for many-valued logics:

Definition 6. *[8] A* `many-valued` *model-theoretic semantics of a given many-valued logic $\mathcal{L}$, with a Gentzen system $\mathcal{G} = \langle \mathbb{L}, \Vdash \rangle$, is the semantic deducibility relation $\models_m$, defined for any $\Gamma = \{s_i = (\Phi_i \vdash \Psi_i) \mid i \in I\}$ and a sequent $s = (\Phi \vdash \Psi) \in \mathbb{L} \subseteq \mathcal{L}_M \times \mathcal{L}_M$ by :*
$\Gamma \models_m s$ iff "all many-valued models of Γ are the models of s", that is,
iff $\forall v \in \mathbb{V}_m(\forall(\Phi_i \vdash \Psi_i) \in \Gamma(\mathfrak{F}(v)(\Phi_i) \leq \mathfrak{F}(v)(\Psi_i)$ implies $\mathfrak{F}(v)(\Phi) \leq \mathfrak{F}(v)(\Psi))$.

Lemma 1. *For any $\Gamma = \{s_i = (\Phi_i \vdash \Psi_i) \mid i \in I\}$ and a sequent $s = (\Phi \vdash \Psi)$, we have that $\Gamma \models_m s$ iff $\forall v \in Mod_\Gamma(\mathfrak{B}(v)(s) = 1)$.*

Proof: We have that $\Gamma \models_m s$ iff
$\forall v \in \mathbb{V}_m(\forall(\Phi_i \vdash \Psi_i) \in \Gamma(\mathfrak{F}(v)(\Phi_i) \leq \mathfrak{F}(v)(\Psi_i)$ implies $\mathfrak{F}(v)(\Phi) \leq \mathfrak{F}(v)(\Psi))$
iff $\forall v \in Mod_\Gamma(\forall(\Phi_i \vdash \Psi_i) \in \Gamma(\mathfrak{F}(v)(\Phi_i) \leq \mathfrak{F}(v)(\Psi_i)$ implies $\mathfrak{F}(v)(\Phi) \leq \mathfrak{F}(v)(\Psi))$
iff $\forall v \in Mod_\Gamma(\mathfrak{F}(v)(\Phi) \leq \mathfrak{F}(v)(\Psi))$
iff $\forall v \in Mod_\Gamma(\mathfrak{B}(v)(s) = 1)$.
□
It is easy to verify that any many-valued logic has a Gentzen-like system $\mathcal{G} = \langle \mathbb{L}, \Vdash \rangle$ (see the definition at the beginning of this section) that is a *normal* logic.

Theorem 1 *Many-valued model theoretic semantics is an adequate semantics for a many-valued logic $\mathcal{L}$ specified by a Gentzen-like logic system $\mathcal{G} = \langle \mathbb{L}, \Vdash \rangle$, that is, it is sound and complete. Consequently, $\Gamma \models_m s$ iff $\Gamma \Vdash s$.*

Proof: Let us prove that for any many valued model $v \in Mod_\Gamma$, the obtained sequent bivaluation $\beta = \mathfrak{B}(v) : \mathcal{L}_M \times \mathcal{L}_M \to \mathbf{2}$ is the characteristic function of the closed theory $\Gamma_v = C(T)$ with $T = \{\Phi \vdash 1,\ 1 \vdash \Phi \mid \Phi \in P_{mm},\ \mathfrak{F}(v)(\Phi) = 1\} \bigcup \{\Phi \vdash 0,\ 0 \vdash \Phi \mid \Phi \in P_{mm},\ \mathfrak{F}(v)(\Phi) = 0\}$.

1. Let us show that for any sequent s, $s \in \Gamma_v$ implies $\beta(s) = 1$:
First of all, for any sequent $s \in T$: if it is of the form $\Phi \vdash 1$ or $1 \vdash \Phi$ (where $\Phi \in P_{mm}$) then $\mathfrak{F}(v)(\Phi) = 1$, thus s is satisfied by v (it holds that $1 \leq 1$ in both cases); if it is of the form $\Phi \vdash 0$ or $0 \vdash \Phi$ then $\mathfrak{F}(v)(\Phi) = 0$, thus s is satisfied by v (it holds that $0 \leq 0$ in both cases). Consequently, *all* sequents in T are satisfied by v.
From Proposition 4, all inference rules in $\mathcal{G}$ are sound w.r.t. the model satisfiability. Thus for any deduction $T \Vdash s$ (i.e., $s \in \Gamma_v$) where all sequents in premises are satisfied by the many-valued valuation (model) v, the deduced sequent $s = (\Phi \vdash \Psi)$ must be satisfied as well, that is, $\mathfrak{F}(v)(\Phi) \leq \mathfrak{F}(v)(\Psi)$, i.e., $\beta(s) = 1$.
2. Let us show that for any sequent s, $\beta(s) = 1$ implies $s \in \Gamma_v$: For *any* sequent $s = (\Phi \vdash \Psi) \in \mathcal{L}_M \times \mathcal{L}_M$ if $\beta(s) = 1$ we have one of the two possible cases:
2.1 Case when $\mathfrak{F}(v)(\Phi) = 0$. Then (from Proposition 2) Φ can be substituted by $\bigvee_{1 \leq i \leq m}(\bigwedge_{1 \leq j \leq m_i}([y_{ij1}]...[y_{ijk_{ij}}])A_{ij})$, i.e., $\mathfrak{F}(v)(\bigvee_{1 \leq i \leq m}(\bigwedge_{1 \leq j \leq m_i}([y_{ij1}]...[y_{ijk_{ij}}]) A_{ij})) = \bigvee_{1 \leq i \leq m} \mathfrak{F}(v)(\bigwedge_{1 \leq j \leq m_i}([y_{ij1}]...[y_{ijk_{ij}}])A_{ij}) = 0$,
that is, for every $1 \leq i \leq m$, $\mathfrak{F}(v)(\bigwedge_{1 \leq j \leq m_i}([y_{ij1}]...[y_{ijk_{ij}}])A_{ij}) = 0$, i.e. $(\bigwedge_{1 \leq j \leq m_i}([y_{ij1}]...[y_{ijk_{ij}}]A_{ij}) \vdash 0) \in T$. Consequently, by applying the inference rule 2b, we deduce $T \Vdash (\bigvee_{1 \leq i \leq m}(\bigwedge_{1 \leq j \leq m_i}([y_{ij1}]...[y_{ijk_{ij}}])A_{ij}) \vdash 0)$, that is, by the substitution inference rule (for $\sigma : \bigvee_{1 \leq i \leq m}(\bigwedge_{1 \leq j \leq m_i}([y_{ij1}]...[y_{ijk_{ij}}])A_{ij}) \mapsto \Phi$) we obtain that $T \Vdash (\Phi \vdash 0)$. From the fact that $0 \vdash \Psi$ is an axiom in $\mathcal{G}$, and by applying the transitive inference rule, we obtain that $T \Vdash (\Phi \vdash \Psi)$, i.e., $s \in C(T) = \Gamma_v$.
2.2 Case when $\mathfrak{F}(v)(\Phi) = 1$. In this case, from the fact that this sequent is satisfied, $\mathfrak{F}(v)(\Psi) = 1$ must be true as well. Thus, we can substitute Ψ by $1 \vee \Psi$, so that we obtain the axiom $1 \vdash 1 \vee \Psi$ in $\mathcal{G}$ and, consequently, by applying the substitution inference rule (for $\sigma : 1 \vee \Psi \mapsto \Psi$), we obtain $T \Vdash (1 \vdash \Psi)$. From the fact that $\Phi \vdash 1$ is an axiom in $\mathcal{G}$ and by applying the transitive inference rule, we obtain that $T \Vdash (\Phi \vdash \Psi)$, i.e., $s \in C(T) = \Gamma_v$.
So, from (1) and (2), we obtain that $\beta(s) = 1$ iff $s \in \Gamma_v$, i.e., the sequent bivaluation β is the characteristic function of a closed set. Consequently, any many-valued *model* v of this many-valued logic $\mathcal{L}$ corresponds to the *closed* bivaluation β which is a characteristic function of a closed theory of sequents: we define the set of all closed bivaluations obtained from the set of many-valued models $v \in Mod_\Gamma$: $Biv_\Gamma = \{\Gamma_v \mid v \in Mod_\Gamma\}$. From the fact that Γ is satisfied by every $v \in Mod_\Gamma$, we have that for every $\Gamma_v \in Biv_\Gamma$, $\Gamma \subseteq \Gamma_v$, so that $C(\Gamma) = \bigcap Biv_\Gamma$ (an intersection of closed sets is a closed set also). Thus, for $s = (\Phi \vdash \Psi)$, $\Gamma \models_m s$ iff
$\forall v \in Mod_\Gamma(\forall(\Phi_i \vdash \Psi_i) \in \Gamma(\mathfrak{F}(v)(\Phi_i) \leq \mathfrak{F}(v)(\Psi_i))$ implies $\mathfrak{F}(v)(\Phi) \leq \mathfrak{F}(v)(\Psi))$
iff $\forall v \in Mod_\Gamma(\forall(\Phi_i \vdash \Psi_i) \in \Gamma(\beta(\Phi_i \vdash \Psi_i) = 1)$ implies $\beta(\Phi \vdash \Psi) = 1)$
iff $\forall v \in Mod_\Gamma(\forall(\Phi_i \vdash \Psi_i) \in \Gamma((\Phi_i \vdash \Psi_i) \in \Gamma_v)$ implies $s \in \Gamma_v)$
iff $\forall \Gamma_v \in Biv_\Gamma(\Gamma \subseteq \Gamma_v$ implies $s \in \Gamma_v)$
iff $\forall \Gamma_v \in Biv_\Gamma(s \in \Gamma_v)$, because $\Gamma \subseteq \Gamma_v$ for each $\Gamma_v \in Biv_\Gamma$
iff $s \in \bigcap Biv_\Gamma = C(\Gamma)$, that is, iff $\Gamma \Vdash s$.
□
Thus, in order to define the model-theoretic semantics for many-valued logics, we do not need to use the matrices: we are able to use only the many-valued valuations and *many-valued models* (i.e., the valuations which satisfy all sequents in Γ of a given many-valued logic $\mathcal{L}$). This point of view is used also for the definition of a new repre-

sentation theorem for many-valued complete lattice based logics in [10,7].
Here, in a many-valued logic $\mathcal{L}$, specified by a set of sequents in Γ, for a formula $\phi \in \mathcal{L}$ that has the same value $x \in X$ (for *any* truth value x) for all many-valued models $v \in Mod_\Gamma$, its modal version $[x]\phi$ is a theorem; that is,
$\forall v \in Mod_\Gamma(v(\phi) = x)$ iff $\Gamma \Vdash (1 \vdash [x]\phi)$,
that corresponds to the *truth-invariance* many-valued entailment (MV) in subsection 1.1. Such a value $x \in X$ does not need to be a designated element $x \in D$, as in the matrix semantics for a many-valued logic. This fact explains way we do not need a semantic specification by matrix designated elements.

Example 2: Let us prove that given an assumption $\Gamma = \{1 \vdash [x]\phi, 1 \vdash [y]\psi\}$ then $[z](\phi \odot \psi)$ for $z = x \odot y$ is deduced from Γ, that is $\Gamma \Vdash ([z](\phi \odot \psi))$; or equivalently if $[x]\phi$ and $[y]\psi$ are valid (i.e., for every valuation $v \in \mathbb{V}_m$, $v : \mathcal{L} \to X$, the values of ϕ and ψ are equal to x and y respectively, i.e. $\forall v \in \mathbb{V}_m(v(\phi) = x$ and $v(\psi) = y))$ then $[z](\phi \odot \psi)$ is valid as well.
As a first step, we introduce the equivalence relation $\equiv$ such that $\Phi \equiv \Psi$ iff $\Phi \vdash \Psi$ and $\Psi \vdash \Phi$ (i.e., Φ iff Ψ). Consequently, from the reflexivity axiom in $\mathcal{G}$, $\Phi \equiv \Phi$, as for example $1 \equiv 1$. The equivalent formulae can be used in the substitution inference rule: if $\Phi \equiv \Psi$ then we can use the substitution of Φ by Ψ, that is, the substitution $\sigma : \Phi \mapsto \Psi$.
Let us show the simple equivalence $\Phi \vee \Phi \equiv \Phi$:
from the reflexivity axiom $\Phi \vdash \Phi$, by using the upper bound inference rule (when $\Psi = \Phi$), we deduce $\Phi \vee \Phi \vdash \Phi$. Then, from the axioms for join $\Phi \vdash \Phi \vee \Phi$, we obtain $\Phi \vee \Phi \equiv \Phi$.
Now, from the assumptions $1 \vdash [x]\phi, 1 \vdash [y]\psi \in \Gamma$, by the lower bound inference rule in $\mathcal{G}$, we obtain (a) $1 \vdash [x]\phi \wedge [y]\psi$, i.e., $\Gamma \Vdash (1 \vdash [x]\phi \wedge [y]\psi)$. Let $z = x \odot y$ and let us denote $[x]\phi \wedge [y]\psi)$ by Φ, so that (a) becomes (a') $1 \vdash \Phi$. Now we can take the axiom for join, (b) $\Phi \vdash \Phi \vee \bigvee_{v,w \in X.v \odot w = z}([v]\phi \wedge [w]\psi)$, so from (a') and (b) and the transitivity rule, we obtain $1 \vdash \Phi \vee \bigvee_{v,w \in X.v \odot w = z}([v]\phi \wedge [w]\psi)$, i.e., (c) $1 \vdash \Phi \vee \Phi \vee \Psi$, where $\Psi = \bigvee_{v,w \in X(v \neq x, w \neq y, v \odot w = z)}([v]\phi \wedge [w]\psi)$. Thus, by substitution $\sigma : \{1 \mapsto 1, \Phi \vee \Phi \mapsto \Phi\}$ and by applying the substitution rule to (c), we deduce the sequent $1 \vdash \Phi \vee \Psi$. That is, (d) $1 \vdash \bigvee_{v,w \in X.v \odot w = z}([v]\phi \wedge [w]\psi)$ (from $\Phi \vee \Psi = \bigvee_{v,w \in X, v \odot w = z}([v]\phi \wedge [w]\psi)$). Consequently, by applying the transitivity rule to the sequent (d) and the introduction axiom $\bigvee_{v,w \in X, v \odot w = z}([v]\phi \wedge [w]\psi) \vdash [z](\phi \odot \psi)$, we deduce $1 \vdash [z](\phi \odot \psi)$, that is $\Gamma \Vdash (1 \vdash [z](\phi \odot \psi))$.
Notice that in such deductions, no value of x, y, z has to be the designated value. We do not make any distinction for the truth values in X.
□
Remark: There is also another way, alternative to 2-valued sequent systems, to reduce the many-valued logics into "meta" 2-valued logics: it is based on Ontological encapsulation [22,18], where each many-valued proposition (or many-valued ground atom $p(a_1, .., a_n)$) is ontologically encapsulated into the "flattened" 2-valued atom $p_F(a_1, .., a_n, x)$ (by enlarging original atoms with a new logic variable whose domain of values is the set of truth values of the complete lattice $x \in X$: roughly, "$p(a_1, .., a_n)$ has a value x" iff $p_F(a_1, .., a_n, x)$ is true). In fact, such a flattened atom is logically equivalent to the sequent $(1 \vdash [x]p(a_1, .., a_n))$.

4 Autoreferential Kripke-style semantics

We are able to define an equivalence relation $\approx_L$ between the formulae of any many-valued logic based on the set of truth values X, in order to define the Lindenbaum algebra for this logic ($\mathcal{L}/\approx_L$), where for any two formulae $\phi, \psi \in \mathcal{L}$, $\quad \phi \approx_L \psi$ iff $\forall v \in \mathbb{V}_m(v(\phi) = v(\psi))$.
Thus, the elements of this quotient algebra $\mathcal{L}/\approx_L$ are the equivalence classes, denoted by $\langle\phi\rangle$.
In particular, we consider an equivalence class $\langle\phi\rangle$ (the set of all equivalent formulae to ϕ w.r.t. $\approx_L$) that has exactly one constant $x \in X$, which is an element of this equivalence class (we abuse a denotation here by denoting a formula such that it has a constant logic value $x \in X$ for every interpretation v by x as well), and we can use it as the representation element for this equivalence class $\langle x\rangle$. Thus, every formula in this equivalence class has the same truth-value as this constant. Consequently, we have the injection $i_X : X \rightarrow \mathcal{L}/_{\approx_L}$ between elements in the complete lattice $(X, \leq)$ and elements in the Lindenbaum algebra such that for any logic value $x \in X$, we obtain the equivalence class $\langle x\rangle = i_X(x) \in \mathcal{L}/_{\approx_L}$. It is easy to extend this injection into a monomorphism between the original algebra and this Lindenbaum algebra, by definition of correspondent connectives in this Lindenbaum algebra. For example: $\langle x \wedge y\rangle = i_X(x \wedge y) = i_X(x) \wedge_L i_X(y) = \langle x\rangle \wedge_L \langle y\rangle$, $\langle\neg x\rangle = i_X(\neg x) = \neg_L i_X(x) = \neg_L\langle x\rangle$, etc..
In an autoreferential semantics [7,8,23], we assume that each equivalence class of formulae $\langle\phi\rangle$ in this Lindenbaum algebra corresponds to one "state - description". In particular, we are interested in the subset of "state - descriptions" that are *invariant* w.r.t. many-valued interpretations v, so that can be used as the possible worlds in the Kripke-style semantics for the original many-valued modal logic. However from the injection i_X, we can take only its inverse image $x = i_X^{-1}(\langle x\rangle) \in X$ for such an invariant "state -description" $\langle x\rangle \in \mathcal{L}/_{\approx_L}$.
Consequently, the set of possible worlds in this autoreferential semantics corresponds to the set of truth values X.
Now we consider the Kripke model (introduced in subsection 1.2) for the 2-valued multi-modal logic language $\mathcal{L}_M$ given in Definition 2, obtained from the many-valued predicate logic language $\mathcal{L}_P$ (defined at the beginning of Section 2):

Definition 7. KRIPKE SEMANTICS*: Let $\mathcal{L}_P$ be a many-valued predicate logic language, based on a set X of truth values, with a set of predicate letters P and Herbrand base H. Let $\mathcal{M}_v = (F, S, V)$ be a Kripke model of its correspondent 2-valued multi-modal logic language $\mathcal{L}_M^*$ with the frame $F = (\mathcal{W}, \{\mathcal{R}_w = \mathcal{W} \times \{w\} \mid w \in \mathcal{W}\})$ where $\mathcal{W} = X \bigcup \mathbf{2}$ and with mapping $\; V : \mathcal{W} \times P \rightarrow \bigcup_{n<\omega} \mathbf{2}^{S^n}$ such that for any n-ary predicate $p \in P$ and tuple $(c_1, .., c_n) \in S^n$, there exists a unique $w \in \mathcal{W}$ such that $V(w,p)(c_1, .., c_n) = 1$.
It defines the Herbrand interpretation $v : H \rightarrow X$ such that $v(p(c_1, .., c_n)) = w$ iff $V(w,p)(c_1, .., c_n) = 1$, and its unique homomorphic extension to all ground formulae $v : \mathcal{L} \rightarrow X$.
Let $\; g : Var \rightarrow S \;$ be an assignment for object variables $\nu_i \in Var$, $i = 1, 2, ..$, and $w \in \mathcal{W}$ then the satisfaction relation $\models$ is defined by $\; \mathcal{M}_v \models_{g,w} \phi \;$ iff $\; v(\phi/g) = w$,*

for any many-valued formula $\phi \in \mathcal{L}_P$. It is extended to all modal formulae in $\mathcal{L}_M^*$ as follows:
1. $\mathcal{M}_v \models_{g,w} 1$ and $\mathcal{M}_v \not\models_{g,w} 0$ for tautology and contradition respectively.
2. $\mathcal{M}_v \models_{g,w} [x]\Phi$ iff $\forall y((w,y) \in \mathcal{R}_x$ implies $\mathcal{M}_v \models_{g,y} \Phi)$ for any $\Phi \in \mathcal{L}_M^*$ or $\Phi \in \mathcal{L}_P$.
3. $\mathcal{M}_v \models_{g,w} \Phi \wedge \Psi$ iff $\mathcal{M}_v \models_{g,w} \Phi$ and $\mathcal{M}_v \models_{g,w} \Psi$ for $\Phi, \Psi \in \mathcal{L}_M^*$.
4. $\mathcal{M}_v \models_{g,w} \Phi \vee \Psi$ iff $\mathcal{M}_v \models_{g,w} \Phi$ or $\mathcal{M}_v \models_{g,w} \Psi$ for $\Phi, \Psi \in \mathcal{L}_M^*$.

Notice that, based on this Kripke model $\mathcal{M}_v$, a many-valued valuation $v : H \to X$ is defined with a unique standard homomorphic extension $v : \mathcal{L} \to X$ as follows (from definition above): for any ground atom $p(c_1, .., c_n) \in H$, we define $v(p(c_1, .., c_n)) = w$ where $w \in X$ is a unique value which satisfies $V(w,p)(c_1, .., c_n) = 1$.
Conversely, given a many-valued model $v : H \to X$ for a many-valued predicate logic language $\mathcal{L}_P$, we define a Kripke model with mapping V such that for any $w \in \mathcal{W}$, n-ary $p \in P$, and a tuple $(c_1, .., c_n) \in S^n$, $V(w,p)(c_1, .., c_n) = 1$ iff $v(p(c_1, .., c_n)) = w$.
Let $\Psi/g \in \mathcal{L}_M$ be a ground formula obtained from $\Psi \in \mathcal{L}_M^*$ by assignment g then we denote the set of worlds where the ground formula $\Psi/g \in \mathcal{L}_M$ is satisfied by $\|\Psi/g\|$, with $\|p(\nu_1, .., \nu_n)/g\| = \{v(p(g(\nu_1), .., g(\nu_n)))\}$ and $\|\phi/g\| = \{v(\phi/g))\}$, $\phi \in \mathcal{L}$.
Thus, different from the original many-valued ground atoms in $\mathcal{L}$, which can be satisfied only in one single world, the modal atoms in $\mathcal{L}_M$ have the standard 2-valued property, that is, they are true or false in these Kripke models and, consequently, are satisfiable in all possible worlds or absolutely not satisfiable in any world. Thus, our positive multi-modal logic with modal atoms $\mathcal{L}_M$ satisfies the classic 2-valued properties:

Proposition 5 *For any ground formula Φ/g of the positive multi-modal logic $\mathcal{L}_M$ defined in Definition 2, $\|\Phi/g\| \in \{\emptyset, \mathcal{W}\}$, where $\emptyset$ is the empty set.*

Proof: By structural induction :
1. $\|1\| = \mathcal{W}$ and $\|0\| = \emptyset$.
2. $\|[x]\phi/g\| = \mathcal{W}$ if $x = v(\phi/g)$; $\emptyset$ otherwise.
Let Φ, Ψ be the two atomic modal formulae such that, by inductive hypothesis $\|\Phi/g\|, \|\Psi/g\| \in \{\emptyset, \mathcal{W}\}$. Then,
3. $\|[x]\Phi\| = \{w \in \mathcal{W} \mid x \in \|\Phi/g\|\} = \mathcal{W}$ if $\|\Phi/g\| = \mathcal{W}$; $\emptyset$ otherwise.
4. $\|(\Phi \wedge \Psi)/g\| = \|\Phi/g\| \bigcap \|\Psi/g\| \in \{\emptyset, \mathcal{W}\}$.
5. $\|(\Phi \vee \Psi)/g\| = \|\Phi/g\| \bigcup \|\Psi/g\| \in \{\emptyset, \mathcal{W}\}$.
Thus, from the fact that any formula $\Phi \in \mathcal{L}_M^*$ is logically equivalent to disjunctive modal formula $\bigvee_{1 \le i \le m}(\bigwedge_{1 \le j \le m_i}([y_{ij1}]...[y_{ijk_{ij}}])A_{ij})$ (from Proposition 2), where each $A_{ij} \in H$ is a ground atom (such that by inductive hypothesis for any modal atom it is true that $\|([y_{ij1}]...[y_{ijk_{ij}}])A_{ij}/g\| \in \{\emptyset, \mathcal{W}\}$), and from points 3 and 4 above, we obtain that
$\|\Phi/g\| = \| \bigvee_{1 \le i \le m}(\bigwedge_{1 \le j \le m_i}([y_{ij1}]...[y_{ijk_{ij}}])A_{ij}/g)\| \in \{\emptyset, \mathcal{W}\}$.
□
The following proposition demonstrates the existence of a one-to-one correspondence between the unique many-valued model of a many-valued logic $\mathcal{L}$ and the Kripke model of a multi-modal positive logic $\mathcal{L}_M$.

Proposition 6 *For any many-valued formula $\phi/g \in \mathcal{L}_P$,*
$v(\phi/g) = x$ *iff* $\mathfrak{F}(v)([x]\phi/g) = 1$ *iff* $\|[x]\phi/g\| = \mathcal{W}$.

Proof: For any ground formula, $\mathfrak{F}(v)([x]\phi/g) = 1$ iff (by Proposition 1) $\mathfrak{F}(v)(\widehat{[x]\phi/g}) = 1$ iff (by Proposition 3) $v(\phi/g) = x$.
Thus, it is enough to prove that $\alpha(\widehat{[x]\phi/g}) = 1$ iff $\|[x]\phi/g\| = \mathcal{W}$, where $\alpha = \mathfrak{F}(v)$.
In the first step, proceeding from left to right, we demonstrate it by structural induction on the length (number of logic connectives) of the formula ϕ :
1. The simplest case when $\phi/g = p(c_1, .., c_k)$, $c_i = g(\nu_i) \in S$, $1 \leq i \leq k$, is a ground atom for the k-ary predicate letter $p \in P$. Then, if $\alpha(\widehat{[x]\phi/g}) = \alpha([x]\widehat{p(c_1, .., c_k)}) = \alpha([x]p(c_1, .., c_k)) = 1$, then (by Proposition 3) $x = v(p(c_1, .., c_k))$, and from Definition 7 we have that $\|[x]\phi/g\| = \mathcal{W}$.
2. Let us now suppose, by inductive hypothesis, that it holds for all formulae with N logical connectives in Σ. Then, for any formula $\phi \in \mathcal{L}_P$ with $N+1$ logical connectives, we have the following two cases:
2.1. Case when $\phi = \sim \phi_1$ where $\sim \in \Sigma$ is a unary connective. Then, if $\alpha(\widehat{[x]\phi/g}) =$ (from Definition 4) $= \alpha(\bigvee_{y \in X. x = \sim y} \widehat{[y]\phi_1/g}) =$ (from Proposition 1) $= \alpha(\bigvee_{y \in X. x = \sim y} [y]\phi_1/g) =$ (from the homomorphism of α) $= \bigvee_{y \in X. x = \sim y} \alpha([y]\phi_1/g) = 1$. Thus, there exists $y \in X$ such that $x = \sim y$ and $\alpha([y]\phi_1/g) = 1$. That is, from Proposition 1, $\alpha(\widehat{[y]\phi_1/g}) = 1$ and from the inductive hypothesis for this y, we obtain (a) $\|[y]\phi_1/g\| = \mathcal{W}$. So, we obtain $\|[x]\phi/g\| = \| \bigvee_{y \in X. x = \sim y} \widehat{[y]\phi_1/g}\| =$ (from the point 5 in Definition 7) $= \bigcup_{y \in X. x = \sim y} \|[y]\phi_1/g\| =$ (from (a)) $= \mathcal{W}$.
2.2. Case when $\phi = \phi_1 \odot \phi_2$, where $\odot \in \Sigma$ is a binary connective.
Then, if $\alpha([x]\widehat{(\phi_1 \odot \phi_2)}/g) =$ (from Definition 4) $= \alpha(\bigvee_{y,z \in X. x = y \odot z}(\widehat{[y]\phi_1/g} \wedge \widehat{[z]\phi_2/g}))$ $=$ (from Proposition 1) $= \alpha(\bigvee_{y,z \in X. x = y \odot z}([y]\phi_1/g \wedge [z]\phi_2/g)) =$ (from the homomorphism of α) $= \bigvee_{y,z \in X. x = y \odot z}(\alpha([y]\phi_1/g) \wedge \alpha([z]\phi_2/g)) = 1$. Then, there exist $y, z \in X$ such that $x = y \odot z$ with $\alpha([y]\phi_1/g) = 1$ and $\alpha([z]\phi_2/g) = 1$. That is, from Proposition 1 $\alpha(\widehat{[y]\phi_1/g}) = 1$, $\alpha(\widehat{[z]\phi_2/g}) = 1$, and from inductive hypothesis for this y we obtain $\|[y]\phi_1/g\| = \mathcal{W}$ and $\|[z]\phi_2/g\| = \mathcal{W}$, that is (b) $\|[y]\phi_1/g \wedge [z]\phi_2/g\| = \|[y]\phi_1/g\| \bigcap \|[z]\phi_2/g\| = \mathcal{W}$. So, we obtain $\|[x]\phi/g\| = \| \bigvee_{y,z \in X. x = y \odot z}(\widehat{[y]\phi_1/g} \wedge \widehat{[z]\phi_2/g})\| =$ (from the point 5 in Definition 7) $= \bigcup_{y,z \in X. x = y \odot z} \|[y]\phi_1/g \wedge [z]\phi_2/g\| =$ (from (a)) $= \mathcal{W}$.
Consequently, we have shown that $\alpha(\widehat{[x]\phi/g}) = 1$ implies $\|[x]\phi/g\| = \mathcal{W}$. Conversely, the proof from right to left is analogous.
Let us now show that for any ground formula $\Phi/g \in \mathcal{L}_M$, $\alpha(\Phi/g) = 1$ iff $\|\Phi/g\| = \mathcal{W}$: From the fact that any $\Phi \in \mathcal{L}_M$ is logically equivalent to disjunctive modal formula $\bigvee_{1 \leq i \leq m}(\bigwedge_{1 \leq j \leq m_i}([y_{ij1}]...[y_{ijk_{ij}}])A_{ij})$, we obtain that if
$\alpha(\Phi/g) = \alpha(\bigvee_{1 \leq i \leq m}(\bigwedge_{1 \leq j \leq m_i}([y_{ij1}]...[y_{ijk_{ij}}])A_{ij}/g)) =$
$= \bigvee_{1 \leq i \leq m}(\bigwedge_{1 \leq j \leq m_i} \alpha(([y_{ij1}]...[y_{ijk_{ij}}])A_{ij}/g)) = 1$
then there exists i ($1 \leq i \leq m$) such that for all $1 \leq j \leq m_i$, $\alpha(([y_{ij1}]...[y_{ijk_{ij}}])A_{ij}/g) = 1$, i.e., $\|([y_{ij1}]...[y_{ijk_{ij}}])A_{ij}/g\| = \mathcal{W}$. Thus, $\| \bigwedge_{1 \leq j \leq m_i}([y_{ij1}]...[y_{ijk_{ij}}])A_{ij}/g\| = \bigcap_{1 \leq j \leq m_i} \|([y_{ij1}]...[y_{ijk_{ij}}])A_{ij}/g)\ \| = \mathcal{W}$ and consequently $\|\Phi/g\| = \mathcal{W}$ and vice

versa.
□

Corollary 1 *Given a many-valued model* v *for a many-valued logic language* $\mathcal{L}_P$*, for any ground formula* $\Phi/g \in \mathcal{L}_M$*,* $\mathfrak{F}(v)(\Phi/g) = 1$ *iff* $\|\Phi/g\| = \mathcal{W}$*, (i.e.,* Φ/g *is true in the Kripke model in Definition 7).*

Proof: By structural recursion and by Propositions 5 and 6.
□
From this corollary, we obtain that any true formula $\Phi \in \mathcal{L}_M$ is also true in the Kripke model, and vice versa. That is, the autoreferential Kripke-stile semantics for the multimodal logic $\mathcal{L}_M$, in Definition 7, is *sound* and *complete*.

5 Conclusion

The main goal of this paper is the development of a new *binary* sequent calculi, with truth-invariance entailment, for a many-valued predicate logic language $\mathcal{L}_P$ with a finite set of truth values X, and the definition of Kripke-like semantics for it, that are both sound and complete. We have not used any ordering of truth values in X nor any algebraic matrix with a strict subset of designated truth values. So, from this point of view, it is the most general semantic approach for the many-valued logics. In more specific cases, when the set X is a complete lattice of truth values, there is also another non-matrix based approach (with truth-preserving entailment) presented in [7,10] where the sequent system is based on the lattice poset of truth values.
In comparison with the standard historical approach based on m-sequents, this approach is deterministic in the way that the axiomatic sequent system is uniquely determined by the set of many-valued logical connectives. This approach is more compact and is a particular implementation of standard two-sided sequent systems, where the left side of each sequent is just a single formula as well. Moreover, this approach is not matrix-based and does not need any definition of a subset of designated elements in X. In other approaches, *for each* subjectively defined subset of designated truth values (consider for example the logic with $n = 10^{30}$ truth-values, $X = \{\frac{i}{n} \mid 1 \leq i \leq n\}$), for the same logic language $\mathcal{L}$ and the same semantics for its logical connectives, we obtain a different deductive system. Here this subjectiveness is avoided, based on the generalization of the 2-valued truth-invariance principle for the logic entailment, and the resulting deductive system for a many-valued logic with fixed semantics of its logical connectives is general and uniquely defined as in all cases of the 2-valued logics.
Different from other approaches, we defined a Kripke-like semantics for this many-valued deductive system as well, because our encapsulation of many-valued logic into the 2-valued sequent system is based on the introduction of the finite set of modal operators for each truth value in X. That is, this 2-valued encapsulation is a modal as in [18]. The frame in this autoreferential Kripke semantics, based on the Lindenbaum algebra considerations, is finite and uniquely determined by the set of truth values in X.

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